Quantum-Electrodynamic Corrections to the Electron Charge Value in Metals*

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The quantum-electrodynamic renormalization of the electron's charge when in a metal (Fermi gas of electrons) environment is calculated. Solid-state many-body techniques are employed rather than the usual quantum-electrodynamic approach, so that the analogy to calculating a material's dielectric constant is exhibited. In metals, the renormalized charge value is increased over the vacuum value by $\delta e/e \cong 4\times 10^{-10}\rho$, where ρ is the metal's electron density in $e/(10^{-24}\ \mathrm{cm}^3)$. These corrections are too small to affect the recent measurements of e/h in metals using the Josephson frequency-voltage relation for superconducting junctions, which are accurate to 10^{-6}. Experiments to be done which will compare e/h values in different metals to several orders of magnitude greater accuracy should detect the corrections to e calculated in this paper.

I. INTRODUCTION

RECENT experiments have employed the Josephson voltage-frequency relation in a superconducting junction to make a precision measurement of e/h.[1] This measurement presently yields the most precise value for the fundamental constants of quantum electrodynamics.

It is remarkable that in spite of the complex solid-state environment that electrons see in metals, the measured value of e/h is independent of particular metal to the accuracy of the experiment which is about 2 ppm.[1]

Although it will be difficult in the near future to improve the accuracy of an absolute measurement of e/h, an experimental comparison of e/h in different metals will be done soon to several orders of magnitude greater accuracy.[2]

The charge on free electrons in the vacuum is known to be the renormalized charge that results from starting with a bare charge and shielding that charge by means of polarization of the vacuum state.[3] The vacuum is polarizable, i.e., it has a dielectric constant, because of the existence of Dirac's negative-energy sea of filled electron energy levels.

This paper will concern itself with the question of whether the renormalization of the electron charge is altered by the presence of a density of positive-energy electrons as is found in a typical metal.

Figure 1 schematically shows the situation. A metal (in the Fermi-gas approximation) can be viewed within the context of quantum electrodynamics as a sea of filled electron states which includes both an infinite sea of negative-energy states and the sea of positive-energy states filled up to a Fermi surface.

When a particular charge interacts with another charge or with an external potential via the qth Fourier component of the potential, then the electron gas will shield that interaction by means of background electrons making virtual excitations from filled states of momentum $\mathbf{p}$ to empty states of momentum $\mathbf{p}+\mathbf{q}$. This leads to an effective shielding factor or dielectric constant for the system.

Figure 1 suggests that the positive-energy Fermi gas will inhibit some of the virtual excitations from the negative-energy sea because of the Pauli principle, thus altering the renormalized charge value. This will indeed be the result of our calculations below.

We will find that the charge value in a metal is different from the free-charge value by an amount of order 10^{-9} which varies from metal to metal. Therefore, the experiments to compare e/h in different metals should see differences of that order.

The calculations will be made in the language of the solid-state many-body problem, though they are identical to the Lorentz-invariant quantum-electrodynamic charge-renormalization calculations of Feynman.[3]

II. HAMILTONIAN OF QUANTUM-ELECTRODYNAMIC SYSTEM

The Lorentz-invariant Hamiltonian for a collection of electrons interacting with each other via the electromagnetic field can be written in a form which hides the Lorentz invariance but corresponds more closely to a

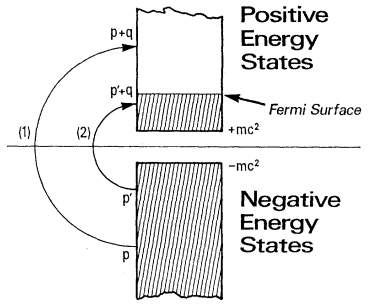

FIG. 1. Complete set of energy levels of an electron gas are shown including the negative-energy sea. Virtual electron excitations (1) shield longitudinal or transverse electromagnetic interactions of wave number $\mathbf{q}$ renormalizing the charge ($e_0 \to e$). Virtual excitations (2) are quenched by the Pauli exclusion principle, leading to a different value for e in an electron gas (metal) environment.

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[1] W. H. Parker, B. N. Taylor, and D. N. Langenberg, Phys. Rev. Letters **18**, 287 (1967).

[2] W. Fairbanks (private communication).

[3] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

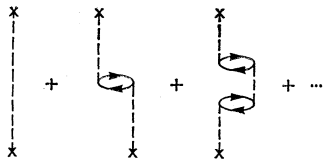

FIG. 2. Instantaneous longitudinal Coulomb interaction is shielded by electron gas polarization bubbles in the RPA. If the dotted line is interpreted to represent the propagation of a transverse photon, the diagram also represents the resulting vacuum polarization which renormalizes the charge value.

solid-state point of view[4]:

$$H=H_f+\frac{1}{2}\int\frac{j_0(\mathbf{x})j_0(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3xd^3x'+\int\mathbf{A}_T(\mathbf{x})\cdot\mathbf{j}(\mathbf{x})d^3x, \quad (1)$$

where H_f is the Hamiltonian for free electrons and the free transverse modes of the electromagnetic field. $j_0(\mathbf{x})$ is the electric charge density, $\mathbf{j}(\mathbf{x})$ is the charge current, and $\mathbf{A}_l(\mathbf{x})$ is the transverse electromagnetic vector potential. In second quantized form (1) reads

$$\sum_{i,\mathbf{p}}\omega_i(p)a_i^\dagger(\mathbf{p})a_i(\mathbf{p})+\sum_{\lambda,\mathbf{p}}pA_\lambda^\dagger(\mathbf{p})A_\lambda(\mathbf{p})$$
$$+2\pi e_0^2\sum_{\mathbf{q},\mathbf{p},\mathbf{p}'}\frac{1}{q^2}[u_i^\dagger(\mathbf{p}+\mathbf{q})]_\gamma[u_j(\mathbf{p})]_\gamma[u_k^\dagger(\mathbf{p}'-\mathbf{q})]_\eta$$
$$\times[u_l(\mathbf{p}')]_\eta a_i^\dagger(\mathbf{p}+\mathbf{q})a_j(\mathbf{p})a_k^\dagger(\mathbf{p}'-\mathbf{q})a_l(\mathbf{p}')$$
$$+\left(\frac{4\pi e_0^2}{2q}\right)^{1/2}\sum_{i,\lambda,\mathbf{p},\mathbf{q}}[u_i^\dagger(\mathbf{p}+\mathbf{q})]_\gamma[\boldsymbol{\alpha}]_{\gamma\eta}[u_j(\mathbf{p})]_\eta\cdot\hat{\boldsymbol{\lambda}}(\mathbf{q})$$
$$\times[A_\lambda^\dagger(-\mathbf{q})+A_\lambda(\mathbf{q})]a_i^\dagger(\mathbf{p}+\mathbf{q})a_j(p). \quad (2)$$

(We have set $\hbar=c=1$.) $A_\lambda^\dagger(\mathbf{q})$ creates a photon of wave vector $\mathbf{q}$, polarization λ. $A_\lambda(\mathbf{q})$ destroys a photon of wave vector $\mathbf{q}$, polarization λ. $a_i^\dagger(\mathbf{p})$ creates an electron of wave vector $\mathbf{p}$, Dirac state i ($i=1, 2, 3, 4$). The four states are positive or negative energy, spin up or down. $a_i(\mathbf{p})$ destroys an electron of wave number $\mathbf{p}$, Dirac state i.

$$[\psi_{i,\mathbf{p}}(x)]_\gamma=e^{i\mathbf{p}\cdot\mathbf{x}}[u_i(\mathbf{p})]_\gamma$$

is the four-component free-Dirac-particle wave function. The subscripts γ and η are to be summed over in equations when appearing twice. The Dirac wave function satisfies the free-particle eigenvalue equation

$$[\boldsymbol{\alpha}\cdot\mathbf{p}c+\beta mc^2]_{\gamma\eta}[\psi_{i,\mathbf{p}}(x)]_\eta=\pm\omega(p)[\psi_{i,\mathbf{p}}(x)]_\gamma,$$

where $[\alpha]_{\gamma\eta}$ are the Dirac 4×4 velocity operator matrices, $[\beta]_{\gamma\eta}$ is the scalar operator matrix, $\hat{\boldsymbol{\lambda}}(\mathbf{q})$ is the unit polarization vector of the photon, and

$$\omega(p)=(m^2+p^2)^{1/2}.$$

[4] P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, London, 1958), 4th ed., Chap. XII.

III. LONGITUDINAL INTERACTION CHARGE RENORMALIZATION

The usual approach to obtaining the shielding of the longitudinal Coulomb interaction in materials is to employ the random-phase approximation (RPA) to sum the polarization bubble processes of Fig. 2 and obtain the dielectric constant of the system. This RPA calculation of the static dielectric constant will be made here, but we will include full account of the negative-energy sea of electrons. From a solid-state physics perspective, the negative-energy sea can be viewed as another band of electrons. From that point of view the vacuum state is an insulator with band gap of $2mc^2$.

The sum of the diagrams in Fig. 2 leads to the expression for the static dielectric constant[5]:

$$\epsilon(\mathbf{q},0)=[1-(4\pi e_0^2/q^2)\Sigma(q)], \quad (3)$$

with

$$\Sigma(q)=\sum_{i,j,\mathbf{p}}\mathrm{Tr}[\Lambda_i(\mathbf{p}+\mathbf{q})\Lambda_j(\mathbf{p})]\frac{C_{ij\mathbf{q}}(\mathbf{p})}{-\omega_{ij\mathbf{q}}(\mathbf{p})}, \quad (4)$$

In (4) we have

$$C_{ij\mathbf{q}}(\mathbf{p})=n_i(\mathbf{p}+\mathbf{q})-n_j(\mathbf{p}),$$

with $n_i(\mathbf{p})$ being the occupation number of the electron state $(i,\mathbf{p})$.

$$\omega_{ij\mathbf{q}}(\mathbf{p})=\omega_i(\mathbf{p}+\mathbf{q})-\omega_j(\mathbf{p}).$$

$[\Lambda_i(\mathbf{p})]_{\gamma\eta}$ is the Dirac free-particle projection operator;

$$[\Lambda_i(\mathbf{p})]_{\gamma\eta}=[u_i(\mathbf{p})]_\gamma[u_i^\dagger(\mathbf{p})]_\eta$$
$$=\frac{\omega(p)+\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m}{2\omega(p)}, \quad \text{summed over positive-energy states}$$
$$=\frac{\omega(p)-\boldsymbol{\alpha}\cdot\mathbf{p}-\beta m}{2\omega(p)}, \quad \text{summed over negative-energy states.}$$

Evaluating the trace in (4) for (ij) representing excitations from negative-energy states into positive-energy states gives in the small q^2 limit

$$\mathrm{Tr}[\Lambda_+(\mathbf{p}+\mathbf{q})\Lambda_-(\mathbf{p})]$$
$$=-q^2\sum_p\frac{1}{2(p^2+m^2)^{3/2}}\left(1-\frac{1}{3}\frac{p^2}{p^2+m^2}\right). \quad (5)$$

Therefore, the dielectric constant is

$$\lim_{q^2\to0}\epsilon(\mathbf{q},0)$$
$$=\left[1+\frac{2}{3}\frac{e_0^2}{\pi}\int\frac{p^2dp}{(p^2+m^2)^{3/2}}\left(\frac{3}{2}-\frac{1}{2}\frac{p^2}{p^2+m^2}\right)\right]. \quad (6)$$

In the vacuum, the sum over p extends over all p and we obtain the well-known quantum-electrodynamic

[5] P. Nozières and D. Pines, Nuovo Cimento **X9**, 470 (1958).

charge renormalization[3]

$$e^2=\frac{e_0^2}{1+(2/3\pi)e_0^2\ln(2\Gamma/m)}, \tag{7}$$

where a momentum cutoff Γ must be introduced to give finiteness.

In the presence of a sea of positive-energy electrons as shown in Fig. 1 the virtual excitations in (5) for $|\mathbf{p}|<p_f$ are quenched because of the Pauli exclusion principle:

$$C_{+-}(\mathbf{p})=n_+(\mathbf{p})-n_-(\mathbf{p})=0, \quad |\mathbf{p}|<p_f.$$

Therefore

$$\lim_{q^2\to 0}\epsilon(q,0)=\left[1+\frac{2}{3\pi}e_0^2\int_{p_f}^{\infty}\frac{p^2dp}{(p^2+m^2)^{3/2}}\left(\frac{3}{2}-\frac{1}{2}\frac{p^2}{p^2+m^2}\right)\right],$$

which gives a renormalized charge value in a metal of

$$e^2(p_f)=e_0^2\Big/\left[1+\frac{2}{3\pi}e_0^2\ln\frac{2\Gamma}{m}-\frac{e_0^2}{3\pi}\left(\frac{p_f}{m}\right)^3\right] \tag{8}$$

for a nonrelativistic electron gas ($p_f\ll mc$). The expression (8) can be expressed solely in terms of finite physical quantities:

$$e^2(p_f)=\frac{e^2}{1-(e^2/3\pi)(p_f/m)^3}. \tag{9}$$

Inserting the necessary $\hbar$ and c factors, (9) yields the fractional change in e due to a Fermi sea of electrons of density ρ:

$$\frac{\delta e}{e}=\frac{1}{6\pi}\left(\frac{e^2}{\hbar c}\right)\left(\frac{p_f}{mc}\right)^3=4.2\times10^{-10}\rho, \tag{10}$$

where ρ is in $e/(10^{-24}\ \mathrm{cm}^3)$. High electron density metals will therefore have a slightly larger value of e than low electron density metals.

If the positive-energy–to–positive-energy virtual electron excitations in Fig. 1 are added to the expression (4) we get the static result

$$\epsilon(q,0)=[e^2(p_f)/e_0^2](1+q_{\mathrm{FT}}^2/q^2)^{-1},$$

with the Fermi-Thomas shielding constant given by

$$q_{\mathrm{FT}}^2=4\pi^{-1}[e^2(p_f)/\hbar^3]mp_f. \tag{11}$$

IV. TRANSVERSE INTERACTION CHARGE RENORMALIZATION

In (1) electrons are coupled to the transverse vector potential, too. This coupling is also renormalized by polarization of the background electrons. We show here that this renormalization is identical to (7) and (9) in the static, $\mathbf{q}=0$ limit.

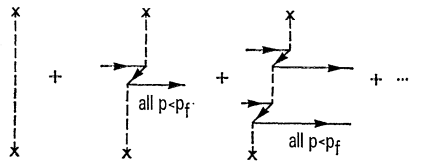

FIG. 3. By conventional Feynman-diagram techniques, the above processes give the corrections to the photon propagator resulting from a Fermi sea of positive-energy electrons. The electron momentum $\mathbf{p}$ is to be summed over all filled states of the electron Fermi gas.

Due to the coupling of photons to the electron current, bare photons are no longer the elementary excitations of our many-body system, even in the vacuum. Instead, we seek a quasiparticle operator

$$P^\dagger(q)=\alpha A_\lambda^\dagger(\mathbf{q})+\beta A_\lambda(-\mathbf{q})+\sum_{i,j,\mathbf{p}}\alpha_{ijq}(\mathbf{p})a_i^\dagger(\mathbf{p}+\mathbf{q})a_j(\mathbf{p}), \tag{12}$$

which creates the physical photon. Using the RPA (also called the linearized equation of motion method) we find $P^+(\mathbf{q})$ such that it obeys[6]

$$[H,P^+(\mathbf{q})]=\Omega P^+(\mathbf{q}). \tag{13}$$

The fact that virtual electron excitations are present in the physical photon is a consequence of the vacuum or background polarization cloud which accompanies a physical photon.

The necessary commutators to solve (13) are obtained from the Hamiltonian (2). For this transverse problem the longitudinal Coulomb interaction term in (2) can be neglected. We will not illustrate all the straightforward steps of solving (13) here. But the following results are obtained:

$$\Omega(\mathbf{q})=\hbar qc;$$

that is, the physical photon has an unaltered energy in the vacuum state. Also,

$$\alpha_{ijq}(\mathbf{p})=(4\pi e_0^2/2q)^{1/2}\hat{\lambda}(\mathbf{q})\times\frac{[u_i^\dagger(\mathbf{p}+\mathbf{q})]_\gamma[\alpha]_{\gamma\eta}[u_j(\mathbf{p})]_\eta}{q-\omega_{ijq}(\mathbf{p})}\alpha. \tag{14}$$

The proper normalization of the physical photon operator is

$$[P(\mathbf{q}),P^\dagger(\mathbf{q})]=1,$$
$$|\alpha|^2+\sum_{i,j,\mathbf{p}}|\alpha_{ijq}(\mathbf{p})|^2C_{ijq}(\mathbf{p})=1. \tag{15}$$

Solving for α gives

$$\alpha^2=\left(1+\frac{2\pi e_0^2}{q}\sum_T(q)\right)^{-1}, \tag{16}$$

[6] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).

with

$$\sum_T(q)=\sum_{i,j,\mathbf{p}}\frac{C_{ijq}(\mathbf{p})}{[q-\omega_{ijq}(\mathbf{p})]^2}\times\mathrm{Tr}[\Lambda_i(\mathbf{p}+\mathbf{q})\boldsymbol{\alpha}\cdot\hat{\boldsymbol{\lambda}}\Lambda_j(\mathbf{p})\boldsymbol{\alpha}\cdot\hat{\boldsymbol{\lambda}}]. \quad (17)$$

In the limit of small q^2, the evaluation of (16) and (17) yields

$$\alpha^2=\left[1+\frac{2}{3\pi}e_0^2\int\frac{p^2dp}{(p^2+m^2)^{3/2}}\left(\frac{3}{2}-\frac{1}{2}\frac{p^2}{p^2+m^2}\right)\right]^{-1}, \quad (18)$$

which is identical to (6).

When two charged particles interact by exchange of a virtual transverse photon, their interaction is diminished by the factor $|\alpha|^2$, which is the reduced probability that a physical photon contains a bare photon.

Hence we conclude that both the longitudinal and transverse renormalization of the electromagnetic charge is given by the same expression. Equation (10) therefore gives the adjustment of the value of e in a metal environment.

If the above calculation were done by conventional Feynman-diagram techniques in quantum electrodynamics, Fig. 3 represents the diagrams which would yield the corrections to e calculated above in this paper.

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Ultrashort-Pulse Generation by Q-Switched Lasers*

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We have undertaken the complete temporal description of pulsed emission by a homogeneously broadened laser, including the effects of spontaneous emission, the detailed geometry of the laser cavity, and the variation of atomic polarization and level populations over wavelength distances. The model is based on traveling-wave equations which are derived from Maxwell's equations and solved in conjunction with boundary conditions imposed at the cavity mirrors. Thus, any direct assumptions concerning the nature of the laser's longitudinal mode structure is avoided. Variations in polarization and population over wavelength distances are treated by means of expansions in spatial Fourier series, having as fundamental a half optical wavelength. The Fourier series are truncated after the first harmonic. The treatment differs from earlier work in that the dephasing of the dipole moment is treated exactly without a rate-equation approximation. Spontaneous emission is simulated both as to spectrum and Gaussian character by including in the dipole equations stochastic shot-noise sources. The model equations are solved numerically, and results include the details of Q-switched pulse evolution from noise for both passive and active switching. In the case of an actively switched laser, the two-photon fluorescence intensity pattern has been calculated. It reveals a well-defined structure of subsidiary intensity maxima, even though subcavities are not assumed in the calculation. The pattern can be correlated directly with the emission pulse structure, and should vary from shot to shot. No single point in the pattern is suitable for a peak-to-background ratio determination. However, if the background is averaged over a distance in the fluorescing medium equal to twice the separation between cavity mirrors, the peak-to-background ratio would be ≈ 1.6, indicating a highly uncorrelated spectrum.

INTRODUCTION

THE emission of radiation by lasers as a succession of pulses, under various conditions, has been a familiar phenomenon for some time.[1–10] Interest in pulsed emission has heightened in recent years as the result of discoveries that solid-state lasers can emit trains of pulses having durations of the order of 10^{-12} sec, not only when mode locking is brought about,[4,5] but even when mode control is not attempted.[9,10] Much of the underlying detail of the resulting pulses cannot be resolved on available oscilloscopes. As a consequence, experimenters have made use of nonlinear optical

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] R. J. Collins, D. F. Nelson, A. L. Schawlow, W. Bond, C. G. B. Garrett, and W. Kaiser, Phys. Rev. Letters **5**, 303 (1960).

[2] L. E. Hargrove, R. L. Fork, and M. A. Pollock, Appl. Phys. Letters **5**, 4 (1964).

[3] M. H. Crowell, IEEE J. Quantum Electron. **QE-1**, 12 (1965).

[4] H. Mocker and R. J. Collins, Appl. Phys. Letters **7**, 270 (1965).

[5] A. J. DeMaria, D. A. Stetser, and H. Heynau, Appl. Phys. Letters **8**, 174 (1966).

[6] F. R. Nash, IEEE J. Quantum Electron. **QE-3**, 189 (1967).

[7] P. W. Smith, IEEE J. Quantum Electron. **QE-3**, 627 (1967).

[8] V. I. Malyshev, A. S. Markin, and A. A. Sychev, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu **6**, 503 (1967) [English transl.: Soviet Phys.—JETP Letters **6**, 34 (1967)]; T. I. Kuznetsova, V. I. Malyshev, and A. S. Markin, Zh. Eksperim. i Teor. Fiz. **52**, 438 (1967) [English transl.: Soviet Phys.—JETP **25**, 286 (1967)].

[9] M. A. Duguay, S. L. Shapiro, and P. M. Rentzepis, Phys. Rev. Letters **19**, 1014 (1967).

[10] M. Bass and D. Woodward, Appl. Phys.Letters **12**, 275(1968).